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DEVELOPMENT IN STRING THEORY

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The string theory is a fast moving subject, both physics wise and in the respect of mathematics. In order to keep up with the discipline it is important to move with new ideas which are being stressed. Here I wish to give extracts from new papers of ideas which I have recently found interesting. There are six papers which are involved:

I. *"Strings formulated directly in 4 dimensions"* A.N. Schellekens
II. *"Remarks on 4D strings"* C.Bachas
III. *"Informal introduction to extended algebras and conformal field theories with c≥1"* F.Ravanini
IV. *"Skein relations and braiding in topological gauge theory"* and *"Modular geometry and the classification of rational conformal field theories"* S.Mukhi
V. *"Chern-Simons theories"* E.Witten
VI. *"Duality and the rôl of nonperturbative effects on the world–sheet"* J.Lauer, J.Mas and H.P. Nilles

I. STRINGS FORMULATED DIRECTLY IN 4-DIMENSIONS

"What is meant by a consistent (closed, fermionic) string theory in d dimensions, is a theory based on a two-dimensional field theory with the following properties:
(i) reparametrization invariance
(ii) conformal invariance
(iii) modular invariance
(iv) world-sheet supersymmetry and superconformal invariance
v) the presence of d right and left-moving scalars (X_R,X_L), **whose zero modes are the space-time coordinates"**.

"The existing ways of satisfying condition (ii) are most easily classified by the left-and right-moving ghost contribution $(c_L c_R)_{\text{ghost}}$ to the central charge of the Virasoro algebra. The possibilities relevant for four dimensions are $(-26,-26)$ (bosonic strings), $(-15,-15)$ (type II strings) and $(-26,-15)$ (heterotic strings). The "matter" fields cancelling these conformal anomalies were traditionally chosen to be 26 bosons (c=26) or ten bosons and ten Majorana–Weyl fermions (c=15)".

Now the art of constructing consistent string theories for d=4 is simply to find the solutions to the conditions listed above, particularly of item (v). The case of d=26 for Bose strings and d=10 for the supersymmetric strings corresponds to the case where ALL the Bose fields in the 2–dimensional underlying theory possess zero modes. This is clearly not necessary and the modern art of constructing consistent theories for d=4 is simply to postulate **only** four scalars (X's) possessing zero modes to correspond to d=4 space–time coordinates.

One of the promising lines of development is to consider internal **orbifolds** for the remaining 6 degrees of freedom in the case of the supersymmetric conformally invariant heterotic theory.

"Orbifolds were first discussed as singular limits of Calabi–Yau manifolds, and later started to lead a life of their own. Their construction has recently been generalized in several ways, by adding background fields ("Wilson lines") or by allowing left–and right–movers to live on different orbifolds ("asymmetric orbifolds")".

"Modular invariant theories (iii) are obtained by twisting boundary conditions of an already modular invariant theory, imposing (at least for Abelian orbifolds) a "level matching" condition to ensure that modular invariance is not destroyed".

It appears that one can construct a number of theories with three families and which preserve the standard model symmetry group $SU_C(3)\times SU_L(2)\times U(1)^n$. The use of Wilson's lines is particularly important in this construction, especially in limiting the number of families.

But even so, there are hundreds of thousands, if not millions, of such theories claimed.

"If all these theories are in fact just different vacua of the same theory, we are still faced with a bewildering choice of vacua. Nevertheless, one should not lose sight of the superiority of string theory over field theory in this respect. In field theory, one can choose arbitrary gauge groups, arbitrary (anomaly−free) representations for all fields, and arbitrary coupling constants. In string theory, one can choose world−sheet boundary conditions. In the space of all possible field theories, the ones that can come from strings are a subset of measure zero. Most of the more exotic Grand Unified Theories that have been proposed in the past cannot come from string theory".

A.N.Schellekens

II. REMARKS ON 4-D STRINGS

A question that often arises is whether the huge number of 4d string models are compactifications of the known 10d superstrings. The answer is yes, no or maybe depending on what one means by "compactification".

Strings seem to evade the wisdom and no−go theorems of traditional Kaluza−Klein compactifications, making the problem of obtaining chirality and enough gauge symmetry in 4d surprisingly easier. Finally, if one asks whether they can be thought of as strings moving on $M_4 \times K_6$, where M_4 is 4d Minkowski space−time K_6 some 6d internal space, the answer is **perhaps,** but to some extent irrelevant. The reason is that strings invalidate our geometric intuition is so many ways, that one is quickly lead to the conclusion that **Riemannian geometry is not the right context for discussing string propagation**. So let us start by illustrating this point with a few examples.

Firstly, strings are extended objects that can wind around a compact space. For instance on a circle of radius R, string states are characterized both by a momentum $p=-\frac{n}{2R}$ quantized in multiples of half the inverse radius, and a winding number $w=mR$ quantized in multiples of the radius. The mass of the state, in appropriate units of order M_{Planck}, is given by:

$$M^2 = \frac{1}{2}\left(\frac{n}{2R} + mR\right)^2 + \Sigma \quad \text{(left oscil. freqs)}$$
$$= \frac{1}{2}\left(\frac{n}{2R} - mR\right)^2 + \Sigma \quad \text{(right oscil. freqs)} \tag{2.1}$$

which shows that for radii of order one, winding is as important as momentum.

One consequence of this is the appearance of new, unexpected from a point-field-theory point of view, massless states. Consider for example the bosonic string compactified on a circle of radius

$$R : M_{26} \rightarrow M_{25} \times S_1(R) \tag{2.2}$$

The massless spectrum contains always the traditional Kaluza-Klein gauge bosons, coming from the reduction of the graviton and antisymmetric tensor:

$$(\partial x \bar{\partial} X^{\mu} \pm \partial X^{\mu} \bar{\partial} x)|0> \quad (\mu = 1,\ldots,25) \tag{2.3}$$

where x is the compactified coordinate. However at the special radius $R=\frac{1}{\sqrt{2}}$, four extra massless charged gauge bosons: $\bar{\partial} X^{\mu}|n = m = \pm 1>$ and $\partial X^{\mu}|n = -m = \pm 1>$ appear. The net result is an enhancement of the gauge symmetry from $U(1) \times U(1)$ to SU(2) × SU(2).

Both the spectrum and the entire S−matrix are unchanged under the duality transformation $R \leftrightarrow \frac{1}{2R}$ which inverts the radius of a circle and interchanges momentum and winding $(n \leftrightarrow m)$. Thus from the string theory point of view, the size of our (nearly) flat universe could equally well be a few Gigasparsecs ($\sim 10^{10}$ light years $\sim 10^{28}$ cm), or a tiny−tiny fraction of an angstrom ($\sim 10^{-88}$Å); in this latter picture a highly energetic proton or electron zooming down an accelarator, would correspond to a string state winding around our tiny universe zillions and zillions of times.

Not only **singularities**, or the **size** of space but even its **dimension** can be an illusion in string theory. This is best illustrated by the SU(2) Wess−Zumino−Witten model, with action:

$$S=k\int\left(G_{ij}^{SU(2)}\eta^{\alpha\beta} + B_{ij}^{SU(2)}\varepsilon^{\alpha\beta}\right)\partial_\alpha X^i \partial_\beta X^j d^2 z \tag{2.4}$$

where $G_{ij}{}^{SU(2)}$ and $H_{ijk}=\frac{\partial B}{\partial X}[ijk]$ are the appropriately normalized metric and, completely antisymmetric 3−form on the SU(2) manifold. This action describes the propagation of a string on a 3−sphere, in the presence of an antisymmetric tensor field background with a Dirac type singularity. What is surprising is that for $k=1$ the entire spectrum of string states and their interactions are identical as those of a string moving on the special circle ($R=\frac{1}{\sqrt{2}}$) that exhibited gauge symmetry enhancement! **Put differently, a 26d bosonic string compactified on a circle is identical to**

a 28d string compactified on a 3−sphere.

The fact that for some sizes we can identify a smooth circle, a singlular orbifold and a three−sphere indicates that Riemannian geometry is totally inadequate for describing the symmetries of even the bosonic string near the Planck scale.

Firstly, as was clearly demonstrated by the heterotic string, left−and right−moving string excitations need not live on the same geomtric space. Secondly, there is world−sheet supersymmetry: at the critical dimension this is implemented by adding to the space−time coordinates X^{μ}, which are world−sheet bosons, superpartners ψ^{μ} which are world−sheet fermions.

2d string world−sheet (not valid for, say, membranes) that there is no intrinsic difference between fermions and bosons. We can thus trade the six internal coordinates $X^5,\ldots X^{10}$ for twelve fermions without destroying world−sheet supersymmetry. Equivalently we may bosonize the six fermions $\psi^5 \ldots \psi^{10}$ and obtain a **nine−dimensional** flat internal space on which strings can move, wind around, etc. It is thanks to this esoteric equivalence between 2d bosons and fermions that one can "evade" a traditional no go statement of Kaluza−Klein compactifications, i.e. one can obtain chirality and **partial breaking of supersymmetry** down to N=1, by what look like simple torus compactifications.

2.1 Conformal field theory (CFT)

What seems at present to replace Riemannian geometry as the proper language for describing string propagation is 2d conformal field theory CFT. For instance, the space of compactifications of the heterotic string down to 4 flat dimensions is the space of all modular invariant CFTs with (1,0) supersymmetry between (left, right) movers and corresponding central charges c=(9,22).

The simplest and hence first CFTs studied in this context are free or Gaussian: toroidal and orbifold compactifications and free fermionic constructions. Despite their simplicity they exhibit an unexpectedly rich structure and illustrate most of the features of strings below the critical dimension: gauge symmetry enhancement and breaking through flat potential directions, chirality and anomaly cancellation, and N=0,1,2 or 4 supersymmetry in 4 dimensional group manifolds, collections of minimal models from the discrete series, and Liouville modes and generalizations. These replace the old constraints of the 70's, i.e. renormalizable interactions and cancellation of anomalies, and will relate in principle the masses and couplings of all string modes. Along the horizontal line, 26th dimension x is compactified on a circle of radius R.

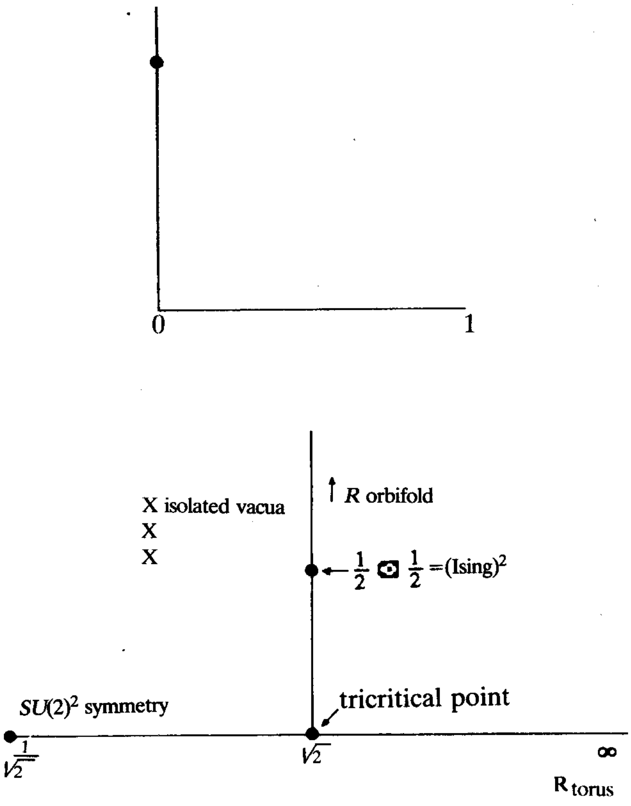


Fig. 1
The phase space of c=1 models.

All of these coexist at: a) the self−dual $SU(2)^2$−symmetric point $R=\frac{1}{\sqrt{2}}$, and b) the "tricritical" point $R_c=\sqrt{2}$, and its dual.

All c=1 models, including the tensor product of two c=½ Ising models, are **Gaussian**, i.e. can be written in terms of a (single) free boson x.

If Gaussian models do not exhaust the phase space of string compactifications down to 4 Minkowski dimensions, we would of course need a more general classification of CFTs.

The prototype is the well known algebraic classification of c<1 (minimal) models. Crossing symmetry and unitarity completely fix the allowed central charges

$$c = 1 \cdot \frac{6}{m(m+1)} \qquad m = 3,4\ldots \tag{2.5}$$

and conformal weights of primary fields:

$$h_{p,q}=\frac{[(m+1)p-mq)^2-1}{4m(m+1)}(1\leq q \leq p< m-1) \tag{2.6}$$

while modular invariance restricts the allowed multiplicities of the latter. Analogous results for $c>1$ have up to now been based on particular extensions of the (conformal) Virasoro algebra.

A rational CFT has a Hilbert space that decomposes into a finite number of representations, under some (unspecified) chiral integer−spin algebra. This is believed to be an inessential restriction. Let $0_1,...,0_M$ be the corresponding primary fields, i.e. highest weight states on which these representations are built, and which we take real for simplicity. Now 3−point interactions are characterized by the **fusion** or **superselection** rules:

N_{ijk}=# of distinct decompositions of $i \otimes j$ into k.

These satisfy the duality condition:

$$\sum_k N_{ijk}N_{k\ell m} = \sum_k N_{i\ell k}N_{kjm} \qquad (2.7)$$

which means that the M $M\times M$ martrices $N_{jk}^{(i)} \equiv N_{ijk}$ commute mutually and, hence, are simultaneously diagonalized by some matrix S. Verlinde's crucial observation is that S is precisely the generator of the modular transformation $\tau \rightarrow 1/\tau$ on the torus, acting on the characters (or partition functions) $X_1...X_M$ of the corresponding representations. This matrix satisfies:

$$S^2=(ST)^3=1 \qquad (2.8)$$

where T is the generator of the $\tau \rightarrow \tau+1$ modular transformation, which only depends on the central charge c and conformal weights $h_1...h_M$ of the O_i:

$$T = e^{-i\pi\frac{c}{12}}\begin{pmatrix} e^{2i\pi h_1} & & 0 \\ & \ddots & \\ 0 & & e^{2i\pi h_M} \end{pmatrix} \qquad (2.9)$$

We thus obtain a very general constraint on the allowed conformal weights and fusion rules, or in the language of strings, the masses and 3−point couplings of particles!

Even if we cannot predict anything new, or "postdict" the parameters of the standard model, it would still be a big step forward to show that these parameters are consistent with the stringy constraints that guarantee the unification of gravity.

It is very easy to construct semirealistic models with, say, one of the standard GUT groups (SU(5), 0(10) or E_6) and enough (≤ 3) chiral families. Consider the ferminonic formulation of 4d heterotic strings. The four space time coordinates X^μ and their left-moving superpartners ψ^μ are supplemented by 44 right fermions η^A and 18 left fermions χ^I, y^I, ω^I (I=1,...6).

2.2 How to obtain the masses of known particles

In terms of M_{Planck} these masses are truly infinitesimal ($\leq 10^{-17}$): they are presumably related to a small breaking of various approximate low −energy symmetries: gauge, chiral and super.

Space−time supersymmetry is much tougher to break. To be precise these are two recent results which seem to exclude classical breaking at scales much less than M_{Planck}: (i) with an assumption of analyticity **continuous** supersymmetry breaking can be excluded in all string compactifications, and (ii) for Gaussian compactifications one can more generally exclude the very existence of a slightly massive gravitino or gaugino, unless some internal radius R becomes huge in units of M_{Planck}^{-1}.

C.Bachas

III. INFORMAL INTRODUCTION TO EXTENDED ALGEBRAS AND CONFORMAL FIELD THEORIES WITH $c \geq 1$

3.1 A classification of all possible CFTs would list all classical vacua of string, as well as all universality classes of two−dimensional critical phenomena. Another reason of this interest lies certainly in the beautiful mathematical structure of CFT. Many areas of Mathematics such as infinite dimensional algebras, Riemann surfaces, monodromy, braid group, etc... are involved and some still mysterious links seem to arise among these structures.

The stress−energy tensor is the generator of conformal transformations in the sense that if we take its Laurent expansion

$$T(z) = \sum_{n\in Z}\frac{L_n}{z^{n+2}} \quad \text{i.e. } L_n = \oint_0 \frac{dz}{2\pi i}T(z)z^{n+1} \qquad (3.1)$$

then the mode L_n generates the infinitesimal conformal transformation $z \longrightarrow z+\varepsilon z^{n+1}$. In particular L_{-1} generates the translations in the z direction, $L_0+\bar{L}_0$ generates the dilations, $L_0 \longrightarrow \bar{L}_0$ the rotations.

In particular the OPE of T with itself reads as

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-2w)^2} + \frac{\partial T(w)}{z-w} + \text{regular terms} \qquad (3.2)$$

The number c is called **conformal anomaly** and plays a central role in the following. Eq. (3.2) implies for the modes of the Virasoro (**Vir**) algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \qquad (3.3)$$

where the constant c appears in the central extension term. We shall use the so called **radial quantization**, where the time is taken as $\log|z|$. In this description $L_0 + \bar{L}_0$ is the Hamiltonian, and its eigenvalues must be bounded below. All the states of a CFT must lie in some irreducible representation of the algebra **Vir** $\otimes$ $\overline{\mathbf{Vir}}$.

These representations are known as **highest weight representations** (HWRs). **Vir** is a rank 2 algebra, i.e. its irreducible representations are labelled by two numbers; for HWRs these are c and Δ

$$(\mathcal{H} = \bigoplus_{\Delta,\bar{\Delta}} \mathcal{N}_{\Delta,\bar{\Delta}} \mathbf{Vir}_c(\Delta) \otimes \overline{\mathbf{Vir}}(\bar{\Delta}) \qquad (3.4)$$

$\mathcal{N}_{\Delta,\bar{\Delta}}$ count the multiplicity of each representation in $\mathcal{H}$, this implies they must always be non-negative integers.

In particular to the HWR there correspond some field $\phi_{\Delta\bar{\Delta}}(z,\bar{z})$ that transform under the conformal group as

$$\phi_{\Delta,\bar{\Delta}} = (z,\bar{z}) = \left(\frac{\partial z'}{\partial z}\right)^{\Delta}\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\Delta} \phi\Delta(z',\bar{z}') \qquad (3.5)$$

They are called **primary fields**. Their OPE with the stress-energy tensor is given by

$$T(z)\phi_{\Delta,\bar{\Delta}}(w,\bar{w}) = \frac{\Delta\phi_{\Delta,\bar{\Delta}}(w,\bar{w})}{(z-w)^2} + \frac{\partial_w \phi_{\Delta,\bar{\Delta}}(w,\bar{w})}{z-w} + \text{regular terms} \qquad (3.6)$$

(called **secondaries**) can be obtained by applying strings of $L_n, n<0$ to $|\Delta\rangle$. The commutation relations imply

$$L_0 L_n^k|\Delta\rangle = (\Delta + nk)L_n^k|\Delta\rangle \qquad (3.7)$$

L_0 eigenvalues organize the space $\mathbf{Vir}_c(\Delta)$ (often called a **module**) so that the states lie on a "stair" whose N-th step (called the N-th **grade**)* has $L_0 = \Delta + N$

states	level	L_0
...	...	...
$L_{-3}\|\Delta\rangle L_{-2}L_{-1}\|\Delta\rangle L_{-1}^3\|\Delta\rangle$	3	$\Delta+3$
$L_{-2}\|\Delta\rangle L_{-1}^2\|\Delta\rangle$	2	$\Delta+2$
$L_{-1}\|\Delta\rangle$	1	$\Delta+1$
$\|\Delta\rangle$	0	Δ

(3.8)

1. for $c<0$ no representation is unitary
2. for $0\leq c<1$ the following set of $\mathbf{Vir}_c(\Delta)$ is unitary

$$c = 1 - \frac{6}{m(m+1)}, \quad m \geq 2, \quad m \in \mathbf{Z} \qquad (3.9)$$

$$\Delta = \Delta_{pq}\frac{[(m+1)p - mq]^2 - 1}{4m(m+1)},$$

$$1 \leq q \leq p \leq m-1 \quad p,q \in \mathbf{Z} \qquad (3.10)$$

3. for $c\geq 1$ all representations are unitary.

Negative Δ's or c is automatically non-unitary. Unitarity is an essential requirement in string theory. Also many statistical systems enjoy it but there are well known cases (percolation, Lee-Yang edge singularity) where unitarity does not hold (i.e. the Hamiltonian is not real).

The OPE algebra of primary fields reads as

$$\phi_i(z,\bar{z})\phi_j(0,0) = \sum_k C_{ij}^k z^{\Delta_k-\Delta_i-\Delta_j}\bar{z}^{\bar{\Delta}_k-\bar{\Delta}_i-\bar{\Delta}_j}[\phi_k(0,0) \qquad (3.11)$$

where now the indices i,j,k run over all primaries of the theory and $[\phi_k(0,0)]$ means contribution from the whole conformal family $[\phi_k]$, which can be seen as an expansion over all secondaries of $[\phi_k]$, whose coefficients are also (in principle) fixed by conformal invariance. The only objects that remain unfixed are the **structure constants** C_{ij}^k. Were these known, one could reduce via iterative applications of OPEs (3.11) all the correlators among primaries to 2 and 3 point functions, which are fixed by projective invariance

$$\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = \delta_{12} z_{12}^{2\Delta}\bar{z}_{12}^{2\bar{\Delta}} \qquad (3.12)$$

$$\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\rangle = C_{12}^3 z_{12}^{\gamma_{12}} z_{13}^{\gamma_{13}} z_{23}^{\gamma_{23}} \bar{z}_{12}^{\bar{\gamma}_{12}} \bar{z}_{13}^{\bar{\gamma}_{13}} \bar{z}_{23}^{\bar{\gamma}_{23}} \qquad (3.13)$$

* We prefer the name **grade** to the other often used **level** to avoid confusion with the level of a Kac-Moody algebra.

where $z_{ab} = z_a - z_b$, $\gamma_{ab} = \Delta_c - \Delta_a - \Delta_b$ with

$$a \neq b \neq c \text{ and } a, b, c = 1, 2, 3.$$

Constraints on $C_{ij}{}^k$ come from the requirement of associativity of the OPE−algebra, which is equivalent to ask **duality** of the 4−point functions So in general C^k_{ij} can be computed if we know the 4−point functions.

3.2 Minimal models exist for $c<1$ only (there is a theorem by Cardy that prevents from the possibility to construct a model with finite number of $\mathbf{Vir}_c(\Delta)$ for $c\geq 1$). More precisely unitary minimal models can exist only for the values of $c=\frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \frac{6}{7}, \cdots$ given by the formula (18) and they can be built up using only the $\mathbf{Vir}_c(\Delta)$ representations such that Δ is contained in the **Kac−table** given by Eq. (3.10). In Fig.2 we give the Kac−tables for the first 3 minimal models m=3,4,5 (m=2 i.e. c=0 is the trivial model containing **1** only). Values of Δ for m=3 give the critical indices of the Ising model, for m=4 those of the tricritical Ising model, for m=5 those of the 5-RSOS model (full table) and of the 3 states Potts model (q−odd lines only).

It is possible to give a complete classification of minimal models through the requirement of modular invariance of the partition function on the torus, which can be constructed mapping the plane into a strip with periodic boundary conditions (i.e. a cylinder) through the conformal transformation $z = \exp(2\pi i w/w_1)$, under which the stress−energy tensor transforms in such a way that

$$L^{cyl}_{-1} = \frac{2\pi i}{\omega_1}\left(L^{pl}_0 - \frac{c}{24}\right), \bar{L}^{cyl}_{-1} = -\frac{2\pi i}{\bar{\omega}_1}\left(L^{pl}_0 - \frac{c}{24}\right) \quad (3.14)$$

i.e. translation on the cylinder are related to dilations on the plane. The constant term $-c/24$ arises as a Casimir effect due to the boundary conditions.

In the case of minimal models $c<1$ a general formula has been given

	1/16
0	1/2

c = 1/2

		1/10
	3/80	3/5
0	7/16	3/2

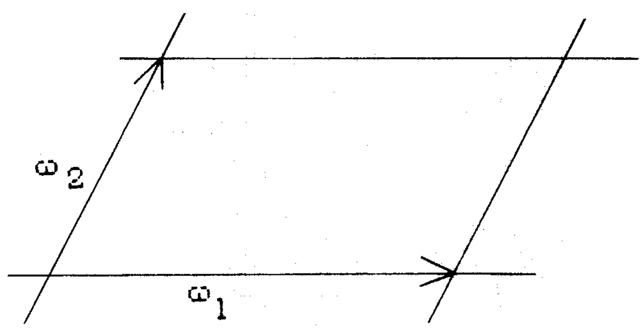
c = 7/10

			1/8
		1/15	2/3
	1/40	21/40	13/8
0	2/5	7/5	3

c = 4/5

Fig.2

Kac−tables for the first 3 minimal models. In these tables the values of Δ_{pq} are reported. p grows horizontally, q grows vertically.



Fig

Dividing the complex plane by the lattice generated by two vectors ω_1, ω_2, (Fig.3) one defines a parallelogram with periodic boundary conditions, i.e. a torus. Think ω_1 as a ***space*** *direction and ω_2 as a* ***time*** *direction.*

$$\chi\Delta = \chi_{p,q}\frac{q^{1/24}}{\prod_{n=1}^{\infty}(1-q^n)}\sum_{k\in Z}\left\{q^{\frac{[2m(m+1)+(m+1)p-mq]^2-1}{4m(m+1)}} - q^{\frac{[2m(m+1)+(m+1)p-mq]^2-1}{4m(m+1)}}\right\} \qquad (3.15)$$

The requirement of modular invariance amounts then to the equations

$$Z(\tau+1) = Z(-1/\tau) = Z(\tau) \qquad (3.16)$$

In the case of minimal models, characters transform as unitary representation of the modular group

$$\chi(\tau+1) = T\chi(\tau), \quad \chi(-1/\tau) = S\chi(\tau) \qquad (3.17)$$

where

$$T_{\Delta,\bar\Delta} - \delta_{\Delta,\Delta} c^{2\pi i}\left(L - \frac{c}{24}\right) \qquad (3.18)$$

$$S_{\Delta,\bar\Delta} = S_{pq,p',q'} = 2\sqrt{\frac{2}{m(m+1)}}(-1)^{(p+q)(p'+q')} \sin\frac{\pi pp'}{m}\sin\frac{\pi qq'}{m+1} \qquad (3.19)$$

CFT are theories with an infinite number of conserved currents. In general theories with infinite number of currents can be integrable, provided the currents carry enough information on the structure of the Hillbert space. The conserved quantities L_n are not yet powerful enough to provide all the information necessary to solve the theory exactly. More currents are needed.

Consider a σ-model on a group manifold **G** with action

$$S_0 = \frac{k}{16\pi}\int d^2x Tr(\partial_a g(x)\partial^a g(x)^{-1}) \qquad (3.20)$$

This model is not conformally invariant, but if we add to the action a Wess–Zumino term

$$S = S_0 + \frac{k}{24}\int d^3y f^{abc} Tr(g^{-1}\partial_a g)(g^{-1}\partial_b g)(g^{-1}\partial_c g) \qquad (3.21)$$

where f_{abc} are the structure constants of **G** and k is an integer, then conformal invariance can be restored as follows. The model (30) enjoys a local invariance **G** $\otimes$ **G** namely $g(z,\bar z)\mapsto \bar\Omega(z)g(z,\bar z)\Omega^\dagger(\bar z), \Omega\in \mathbf{G}, \bar\Omega\in \bar{\mathbf{G}}$ generated by the currents

$$J^a(z) = -\frac{k}{2}(\partial^a g)g^{-1}, \; \bar J^a(\bar z) = -\frac{k}{2}g^{-1}(\partial^a g) \qquad (3.22)$$

One can write down for this invariance Ward identities similar to conformal ones. These in turn imply the following OPE between the currents $J(z)$

$$J^a(z)J^b(w) = \frac{k\delta^{ab}/2}{(z-w)^2} + \frac{f^{ab}_c J^c(w)}{w-z} + \text{regular terms} \qquad (3.23)$$

The number k appearing in the central extension of this algebra is called the **level**. Unitarity as well as consistency of the model requires k positive integer. The modes J^a_n of the currents $J^a(z)$, defined through the expansion $J^a(z) = \sum_{n\in Z}\frac{J^a_n}{z^{n+1}}$, satisfy a $\bar{\mathbf{G}}$ Kac–Moody algebra

$$|J^a_m, J^b_n| = if^{ab}_c J^c_{m+n} + \frac{k}{2}m\delta^{ab}\delta_{m+n,0} \qquad (3.24)$$

The stress–energy tensor can be shown to have the form

$$T(z) = \frac{1}{k+\text{cox}\,\mathbf{G}} \ddagger J_a(z)J_a(z)\ddagger \qquad (3.25)$$

where $\ddagger\ \ddagger$ denotes ordering w.r.t. the modes of $J(z)$. It modes give a **Vir** algebra with

$$c = \frac{k\dim\mathbf{G}}{k+\text{cox}\,\mathbf{G}} \qquad (3.26)$$

Therefore the model is also conformal invariant.

3.3 These models can be thought as minimal models of a conformal algebra enlarged by an internal continuous symmetry **G** $\otimes$ **G**. The $\hat{\mathbf{G}}$ primary fields have dimensions

$$\Delta_p = \frac{\vec p\cdot(\vec p + 2\vec\rho)}{k+\text{cox}\,\mathbf{G}} \qquad (3.27)$$

For the particular case of **G**=$SU(2)$ these formulas simplify to

$$c = \frac{3k}{k+2}, \quad \Delta_l = \frac{l(l+2)}{4(k+2)}, \quad l = 0,1,\dots k \qquad (3.28)$$

For **G**=$SU(2)$ an exhaustive classification of solutions $\mathcal{L}$ is known. This classification is better described in terms of simply–laced simple (A,D,E type) Lie algebras. One can first associated to any solution $\mathcal{L}$ a simply–laced simple Lie algebra whose Coxeter number is equal to k+2.

Consider a free massless boson, compactified on a circle of radius R i.e. a $U(1)\otimes\overline{U(1)}$ abelian σ–model with action

$$S = \frac{1}{2\pi}\int d^2x(\partial_\mu\phi(x))^2 \qquad (3.29)$$

with equation of motion $\partial_z\partial_{\bar{z}}\phi(z,\bar{z})=0$, which admits solutions of the kind $\phi(z,\bar{z})=\varphi(z)+\bar{\varphi}(\bar{z})$. This is CFT with $c=1$, as can be easily seen form the OPE of the stress–energy tensor $T(z)=-½:(\partial_z\varphi)^2:$ with itself, that can be easily calculated using Wick theorem and $\langle\varphi(z)\varphi(0)\rangle=\log z$. The equation of motion implies that the spin 1 current $J(z)=i\partial_i\varphi(z)$ (and its right analog $\bar{J}(\bar{z})=i\partial_z\bar{\varphi}(\bar{z})$) is conserved. The vertices (primary fields)

$$V_a(z) =: e^{ia\varphi(z)}: \qquad (3.30)$$

with $U(1)$ charge (i.e. eigenvalue of a_0) α and conformal dimension $\Delta_a = a^2/2$ They have the product rule

$$:e^{ia\varphi(z)}::e^{i\beta\varphi(0)}:= z^{2\alpha\beta}:e^{i\sigma\varphi(z)+i\beta\varphi(0)}: \qquad (3.31)$$

The modular invariant partition function on the torus can be written directly from the action

$$Z(R) = \frac{(q\bar{q})^{1/24}}{\prod_{k=1}^{\infty}(1-q^k)(1-q^k)} \sum_{n,m=-\infty}^{+\infty} q^{1/2(\frac{m}{2\pi}+nR)^2}\bar{q}^{1/2(\frac{m}{2\pi}-nR)} \qquad (3.32)$$

that shows the duality $Z(R)=Z(1/2R)$. At the self–dual value $R=1/\sqrt{2}$ two new operators of conformal dimensions $(\Delta,\bar{\Delta})=(1,0)$ (and two right analogs $(0,1)$) appear, thus enlarging the symmetry algebra to $\widehat{SU}(2)$

$$J_\pm(z) =: e^{\pm i\sqrt{2}\varphi(z)}: \qquad (3.33)$$

In fact this point coincides with the $SU(2)_{k=1}$ WZW model, which has indeed $c=1$. We can also construct an orbifold of the bosonic theory, identifying $\varphi\equiv-\varphi$. It can be shown that the partition function on the orbifold line (which describes the critical line of the Ashkin Teller model) can be written as

$$Z_{orb}(R) = \frac{1}{2}\left[Z(R) + 2Z(\sqrt{2}) - Z\left(\frac{1}{\sqrt{2}}\right)\right] \qquad (3.34)$$

One can ask if it is possible to construct algebras that combine conformal symmetry with global discrete symmetries, for example with $\mathbf{Z}_N$ symmetries.

Analogously, we can define similar variables in the $\mathbf{Z}_3$ models, the well known 3–state Potts model, by considering the products $\sigma\mu \sim \psi, \sigma^\dagger, \mu^\dagger \sim \psi^\dagger$, $\sigma\mu^\dagger\bar{\psi}$ and $\sigma^\dagger\mu \sim \bar{\psi}^t$, ψ and $\psi^\dagger$ have conformal dimension $(\frac{2}{3},0)$ and $\bar{\psi},\bar{\psi}^\dagger$ have $(0,\frac{2}{3})$. It can be shown in general that any conformal field ϕ of the type $(\Delta,0)$ satisfies $\partial_{\bar{z}}\phi=0$ (and conversely if $\bar{\phi}$ is of type $(0,\Delta)$ then $\partial_z\bar{\phi}=0$). So $\psi,\psi^\dagger$ (and $\bar{\psi},\bar{\psi}^\dagger$ for the right part) are conserved currents of spin $\frac{2}{3}$, called **parafermions**.

More generally, we can define parafermions for any $\mathbf{Z}_N$ model, and construct an algebra for each case.

There is a remarkable link between $SU(2)_k$ WZW model and $\mathbf{Z}_k$ parafermionic theories that allows to compute the correlation functions of the latter. Namely, if we take a free boson φ and a $\mathbf{Z}_k$ parafermion ψ_1 not mutually interacting, the currents

$$J_+(z) = \psi_1(z):e^{i\sqrt{\frac{2}{k}}\varphi(z)}:$$
$$J_-(z) = \psi_1^\dagger(z):e^{-i\sqrt{\frac{2}{k}}\varphi(z)}:$$
$$J_0(z) = \sqrt{k}\partial_z\varphi(z) \qquad (3.35)$$

generate a $SU(2)$ Kac–Moody algebra at level k, as can easily be seen from OPEs of the boson vertices and of parafermions.

We conclude that the following decomposition holds

$$\boxed{\begin{matrix}SU(2)_L\\ \text{WZW model}\end{matrix}} = \boxed{\begin{matrix}\mathbf{Z}_L\\ \text{parafermion}\end{matrix}} \otimes \boxed{\begin{matrix}\text{free}\\ \text{boson}\end{matrix}} \qquad (3.36)$$

One can also generalize this construction to every Lie algebra $\mathbf{G}$ and to rank $\mathbf{G}$ free bosons:

$$\boxed{\begin{matrix}\mathbf{G}_k\\ \text{WZW model}\end{matrix}} = \boxed{\begin{matrix}\mathbf{G}_k \text{ Gepner}\\ \text{parafermion}\end{matrix}} \otimes \boxed{\begin{matrix}\text{rank } \mathbf{G} \text{ dimensional}\\ \text{free boson}\end{matrix}} \qquad (3.37)$$

thus defining new parafermionic theories with more complicate abelian symmetries. These models and their extended algebras are presented.

Therefore we are "conquering" some points in the CFT space of theories, above the wall $c=1$ established by Cardy theorm.

It is possible to construct a lot of CFTs starting from the WZW models. Given a certain simple Lie algebra $\mathbf{G}$, we can consider its Kac–Moody extension $\hat{\mathbf{G}}$ and build up a model out of the HWRs of $\hat{\mathbf{G}}$ at a certain level k, namely the already described WZW model.

coset	algebra	models	c
$SU(2)_N$	$\widehat{SU}(2)$	SU(2) WZW	$\frac{3N}{N+2}$
G_N	$\hat{\mathbf{G}}$	G WZW	$\frac{Nd}{N+Q}$
$\frac{SU(2)_N \otimes SU(2)_1}{SU(2)^{diag}_{N+1}}$	**Vir**	minimal	$1-\frac{6}{m(m+1)}$
$\frac{SU(2)_N \otimes SU(2)_2}{SU(2)^{diag}_{N+2}}$	**Sca**	N=1 Susy	$\frac{3}{2}(1-\frac{8}{m(m+2)})$
$\frac{SU(2)_N \otimes U(1)}{U(1)^{diag}}$	**Sca**$_2$	N=2 Susy	$3(1-\frac{2}{m})$
$\frac{SU(2)_N \otimes SU(2)_4}{SU(2)^{diag}_{N+4}}$	$\hat{\mathbf{S}}_3$	$\mathbf{S}_3$-symm.	$2(1-\frac{12}{m(m+4)})$
$\frac{SU(2)_N \otimes SU(2)_L}{SU(2)^{diag}_{N+L}}$	$\mathcal{A}_L$	coset	$\frac{3L}{L+2}(1-\frac{2(L+2)}{m(m+L)})$
$\frac{SU(2)_N}{U(1)}$	$\hat{\mathbf{Z}}_N$	$\mathbf{Z}_n$-parafer.	$\frac{2(N-1)}{N+2}$
$\frac{\mathbf{G}_N}{U(1)^r}$	$\hat{\mathbf{Z}}_G$	Gepner paraf.	$\frac{Nd}{N+Q}-r$
$\frac{SU(3)_N \otimes SU(3)_1}{SU(3)^{diag}_{N+1}}$	$\mathbf{W}_3$	$\mathbf{Z}_3$-symm.	$2(1-\frac{12}{m(m+1)})$
$\frac{SU(n)_N \otimes SU(n)_1}{SU(n)^{diag}_{N+1}}$	$\mathbf{W}_n$	$\mathbf{Z}_n$-symm.	$(n-1)(1-\frac{n(n+1)}{m(m+1)})$
$\frac{\mathbf{G}_N \otimes \mathbf{G}_L}{\mathbf{G}^{diag}_{N+L}}$	$\mathbf{W}^G_L$	coset	$\frac{Ld}{L+Q}(1-\frac{Q(Q+L)}{m(m+L)})$

Table 1

Some GKO constructions and the corresponding ***Ext*** *algebras and coset models. Here Q=cox* ***G****, d=dim* ***G****, r=rank* ***G*** *and $m=N+Q$.*

Theorem 1 If $\mathbf{G}=\mathbf{A}\otimes\mathbf{B}$ then the quantity $T(G_{k,l}) = T(\mathbf{A}_k) + T(\mathbf{B}_l)$ satisfies a **Vir** algebra and therefore can be considered as the stress–energy tensor of a conformal field theory with $c(\mathbf{G}_{k,l}) = c(\mathbf{A}_k) + c(\mathbf{B}_l)$

Theorem 2 If **H** is a sub–algebra of **G**, then the quantity $T(\mathbf{G}_k/\mathbf{H}_l)=T(\mathbf{G}_k) - T(\mathbf{H}_l)$ also satisfies a **Vir** algebra with $c(\mathbf{G}_k/\mathbf{H}_l) = c(\mathbf{G}_k) - c(\mathbf{H}_l)$. Further $T(\mathbf{G}_k/\mathbf{H}_l)$ commutes with $T(\mathbf{H})$ and with all the $\mathbf{H}_l$Kac–Moody currents.

These two theorems give the possibility to construct CFTs starting from any coset space **G/H** (Table 1).

3.4 Free boson with screening charge: the Feigin-Fuchs (FF) construction

Another very important tool in the research on CFTs for both $c<1$ or $c\geq 1$ is the celebrated FF construction.

Let us add to the free boson stress energy tensor a total derivative term (that does not change the dynamical properties of the model, but only the boudnary conditions at infinity). This can be done adding a second derivative term to the stress–energy tensor of the free boson, i.e. turning on a so-called screening charge at infinity α_0

$$T(z) = -\frac{1}{2} : (\partial_z \varphi)^2 : + i\alpha_0 \varphi_z^2 \varphi \qquad (3.38)$$

Therefore this deformation of the free bosonic theory is still a CFT. The central charge is modified from $c=1$ to $c=1 - 24\,\alpha_0^2$. By suitable choosing

$$a_0 = \frac{1}{2\sqrt{m(m+1)}} \qquad (3.39)$$

one can reproduce the formula for the c values for the minimal series. To realize primary fields, we use vertex operators $V_\alpha(z) =: e^{i\alpha\varphi(z)}$ having conformal dimensions $\Delta_\alpha = \frac{1}{2}\alpha(2\alpha_0 - \alpha) = \Delta_{2\alpha_0 - \alpha}$ Any correlator $\langle V_{\alpha_1}, \dots, V_{\alpha_n}\rangle$ of the free bosonic theory is different from zero only if the $U(1)$ charge is conserved: $\sum_{i=1}^{n} \alpha_i = 0$ When we turn on the screening charge this is modified to

$$\sum_{i=1}^{n} \alpha_i = 2\alpha_0 \qquad (3.40)$$

In particular consider the 4–point correlator

$$\langle V_\alpha V_\alpha V_\alpha V_{2\alpha_0 - \alpha}\rangle \qquad (3.41)$$

It never satisfies condition (3.40). On the other hand this correlator must be non–zero as V_α and $V_{2\alpha_0-\alpha}$ represent the same operator. To solve this contradiction we introduce the so called **screening operators** i.e. operators of conformal dimension zero, so that they do not change the conformal properties of the correlation functions when inserted in them, but can restore the charge conservation (3.40). These operators have the integral form

$$S = \oint_C dz V_\beta(z) \qquad (3.42)$$

Since S are required to have conformal dimension 0, dz has dimension -1, V_β must have dimension 1, hence β must be chosen to satisfy the equation $\frac{1}{2}\beta(\beta - 2\alpha_0) = 1$, which has two solutions

$$a_\pm = \frac{1}{2}(\alpha_0 \pm \sqrt{a_0^2 + 1}) \qquad (3.43)$$

Suppose now to insert in (3.41) $p-1$ operators of type V_{α_+} and $q-1$ of type V_{α_-}. Eq. (3.40) is modified by these insertions to

$$2\alpha + (p-1)\alpha_+ + (q-1)\alpha_- = 0 \qquad (3.44)$$

This constraints Δ to take the values

$$\Delta = \Delta_{pq} = -\alpha_0^2 + \left[\frac{p}{2}\alpha_+ + \frac{q}{2}\alpha_-\right]^2 \qquad (3.45)$$

Inserting for α_0 its value (3.39) we get Eq. (33.10). Even further, one can generalize the construction including the Gepner parafermions

$\mathbf{G}_L$ Gepner parafermion	$\otimes$	rank **G** dimensional screened boson

$$= \boxed{\frac{G_N \otimes G_L}{G^{diag}_{N+L}} \text{ coset models}} \qquad (3.46)$$

All the minimal models of **Ext**–algebras as well as GKO and FF models, have the fundamental property that their Hilbert spaces are finitely reducible in some set of quantum numbers i

$$\mathcal{H} = \bigotimes_{i(finite)} \mathcal{N}_i U_i \otimes \bar{U}_i \qquad (3.47)$$

where $U_i, \bar{U}_i$ are some (generally reducible) representations of **Vir**. We formalize this property calling **rational** CFT (RCFT) every theory that enjoys it.

Theorem 3 For each RCFT, blocks always satisfy an ordinary differential equation of degree N, fuchsian at 0,1,∞ and at a finite set of other "apparent" singularities.

Let $f_1 \dots f_n$ be the blocks: every 4–points correlator f must be a linear combination of them (forget the x part, i.e. consider it as part of the coefficient)

$$f = \sum_{r=1}^{N} \lambda_r f_r \qquad (3.48)$$

Consider the determinant

$$W_0(x) = \det = \begin{bmatrix} f & f_1 & f_2 & \dots & f_N \\ \partial f & \partial f_1 & \partial f_2 & \dots & \partial f_N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial^{N-1} f & \partial^{N-1} f_1 & \partial^{N-1} f_2 & \dots & \partial^{N-1} f_N \\ \partial^N f & \partial^N f_1 & \partial^N f_2 & \dots & \partial^N f_N \end{bmatrix} \qquad (3.49)$$

The first column in W_0 is a linear combination of the others, so clearly $W_0=0$.

On the other hand, one can compute W_0 with Laplace rule, and get

$$(-1)^N W_N \partial^N f + (-1)^{N-1} W_{N-1} \partial^{N-1} f + \ldots$$

$$+ W_1 f = 0 \qquad (3.50)$$

where K_k stands for the minor of W_0 where the $k-$th row and the first column have been deleted. If we divide now this relation by W_N we get

$$\partial^N f - \frac{W_N - \mathbf{1}}{W_N} \partial^{N-1} f + \cdots + (-1)^N \frac{W_1}{W_N} f = 0 \qquad (3.51)$$

i.e. an ordinary differential equation to be satisfied by the 4−point correlators. The singularities in this differential equations can be at $0, 1, \infty$ (physical singularities) and at any other point x_a where $W_N(x_a) = 0$. The physical singularities must be of the fuchsian type, otherwise the behaviour of the solutions cannot be holomorphic times branch singularity, thus preventing from the interpretation of these solutions as blocks. The singularities at x_a must be apparent, i.e. singularities of the differential equation but not of its solutions. This means that the exponent at these must be positive integers.

Two consistent Fusion Algebras are possible:

1. $\phi \times \phi = 1$

In this case one has $N=1$ and it is easy to compute $\alpha = -2\Delta, \beta = -2\Delta, \gamma = 0$ because the exchanged field is $\phi_{p1} = 1$ in each channel $p = s,t,u$ and $\Delta_{p1} = 0$. We obtain $-4\Delta = -R$ thus yielding $\Delta = R/4$. We have the following identifications with known models:

$$R = 1 \quad \Delta = \frac{1}{4} \quad c = 1 \rightarrow SU(2)k = 1WZW$$

$$R = 3 \quad \Delta = \frac{3}{4} \quad c = 7 \rightarrow E_7 = 1WZW \qquad (3.52)$$

2. $\phi \times \phi = 1 + \phi$

The same procedure as above (now $N=2$ and $\Delta_{p1} = 0, \Delta_{p2} = \Delta$ in each channel $p = s,t,u$) gives the restriction $\Delta = (R-1)/5$ and we can predict in this case that the lowest value of Δ is $-1/5$. We identify the following known models

$$R = 0 \quad \Delta = -\frac{1}{5} \quad c = -\frac{22}{5} \rightarrow \text{Lee} - \text{Yang singularity}$$

$$R = 3 \quad \Delta = \frac{2}{5} \quad c = \frac{14}{5} \rightarrow G_2 k = 1WZW$$

$$R = 4 \quad \Delta = \frac{3}{5} \quad c = \frac{26}{5} \rightarrow F_4 k = 1WZW \qquad (3.53)$$

Theorem 4 The matrix S diagonalizing the fusion rules is the same that transforms characters under the $\tau \rightarrow -1/\tau$ transformation of the modular group. Hence S must satisfy $S^2 = C$ and $(ST)^3 = 1$, i.e. the relations defining the modular group. Let us stress how surprising this result is: FAs are something related to OPEs, i.e. something local. Modular invariance is dictated by boundary conditions., i.e. it is something global. A mysterious link between local and global physics arises in CFT!

The constraint $S^2 = C$ is not satisfied by all fusion rules: it acts as a strong selector that tells us that only a subset of FA are good for CFT. The other constraint $(ST)^3 = 1$ puts constraints on c and Δ that become particularly powerful when combined with the previously mentioned ones. For example one could consider FAs of the type $\phi \times \phi = 1 + n\phi$ for all $n = 0,1,2,\ldots$ If we combine these constraints and those from $(ST)^3 = 1$, we discover that the two are compatible for $n = 0,1$ only. All the FAs of this type with $n \geq 2$ are inconsistent.

1. For **Abelian groups** on the r.h.s. of the fusion rule it appears only one representation. Hence all "one operator" (having only one field on the r.h.s.) are exhaustively classified: they are in one to one correspondence with finite abelian groups.

2. For **non abelian groups** we have the partial result that for all the cases examined so far $S^2 \neq C$ so that it seems that all these FAs do not give (if not enlarged by introduction of some other operator) consistent CFTs. Is this rule general? In this case it would provide a very strong selection rule: all FAs that can be constructed from a non−abelian finite group are inconsistent. But this is far to be proven.

3. Not all FAs are generated by finite groups. As a counterexample we can give the Ising fusion algebra.

4. There are very recent links with the **Theory of Graphs.**

There are a lot of other issues in this rapidly developing field, such as the link to two−dimensional quantum gravity and the perturbation of CFT to go off−criticality, or the link with 3−dim. Chern−Simons gauge theories. What arises from the present research is that CFT has a very deep and beautiful mathematical structure. We think this reason would be enough to continue studying it.

F. Ravanini

IV. ALGEBRAIC ASPECTS OF TOPOLOGICAL GAUGE THEORIES

4.1 Skein Relations and Braiding in Topological Gauge Theory

We derive the skein relations in Chern−Simons

Witten (CSW) topological gauge theory in three dimensions, for arbitrary integrable representations of $SU(2)_k$. These are of order $N=2_j+1$ for braids of isospin $_j$, generalizing the usual quadratic relations.

Recently, another remarkable relation of knots and links to physics has been found. It has been shown that in a three–dimensonal gauge theory with Chern–Simons action, the expectation values of Wilson loops are polynomial invariants for the corresponding knots. The relationship of this theory (which we henceforth refer to as Chern–Simons–Witten (CSW) theory) to rational conformal field theory (RCFT) has also been discussed in these papers. In particular that the braiding matrix for conformal blocks in RCFT is given by the expectation value in CSW theory, on S^3, of a tetrahedral Wilson graph, suitably framed. In what follows, we always deal with CSW theory defined on S^3, although the theory on other 3–mainfolds can also be studied using surgery.

In the knot theory literature, quadratic skein relations are the key to defining tand computing polynomial invariants, since they allow one to systematically "unknot" knots and links *In* $SU(n)_k$ *CSW theory with* lines in the representation n, the braiding matrix has been computed using the skein relation, and this can be used in a transfer matrix formalism to evaluate the expectation value of any Wilson loop.

4.2 Modular Geometry and the Classification of Rational Conformal Field Theories

If we study all CFTs (Conformal Field Theories) with finite number of characters, the method involves writing the most general modular-invariant differential equation on the moduli space of the torus, and looking for solutions which satisfy the axioms of conformal field theory.

Rational conformal field theories (RCFT) were discovered by Belavin, Polyakov and Zamolodchikov (BPZ), although the term "rational" was used in this context later. These authors constructed a class of RCFT with Virasoro central charge $c<1$, the so-called minimal series. Their procedure starts from highest-weight representations of the Virasoro algebra, and the existence of null-vectors in these representations. The fields creating highest-weight states are called primaries, and their descendants under the Virasoro algebra, secondaries, It is shown that the constraints due to the presence of null vectors are strong enough to deduce the primary field content of the theory.

In this sense, the $c<1$ RCFT's are completely classified and exactly solved on the plane.

The situation is rather different for $c>1$ or for Riemann surfaces other than the plane.

As to the classification of RCFT, most of the progress in the last few years involved finding representations of extended chiral algebras which include the Virasoro algebra as a proper subalgebra. Knizhnik and Zamolodchikov studies RCFT based on affine Kac-Moody algebras associated to compact simple Lie groups.

This procedure was also extended to superconformal, parafermion and W-algebras, among others.

Unfortunately, none of these procedures provided a general method to classify and construct all rational conformal field theories.

Two important, and related inputs became available more recently. One was the observation, by Friedan and Skenker, that the characters of RCFT can be thought of as holomorphic sections of a certain line bundle over moduli space. The other was the remarkable discovery, by Verlinde, that the fusion rules are diagonalized by the matrix S_{ij} which implements the modular transformation $\tau \rightarrow -\frac{1}{\tau}$ on the characters. Verlinde's result implies certain constraints on the values of the central charge and the conformal dimensions in an RCFT, given the number of characters and the fusion rules.

An important ingredient that was lacking in approaches to RCFT based on "modular geometry" was the fact that, besides forming a section of a line bundle on moduli space, the characters have another important property: their power-series expansion in the variable $q \equiv \exp(2\pi i r)$ involves coefficients which are positive integers. These integers count the number of secondaries at a given level, with respect to whatever chiral algebra is involved.

The characters of a conformal field theory are defined by

$$\chi_i(\tau) \equiv tr_i \quad q^{L_0 - \frac{c}{24}}, \quad q \equiv e^{2\pi i\tau} \qquad (4.1)$$

where L_0 is the zero mode operator in the Virasoro algebra, c is the Virasoro central charge, and the trace is taken over all states above a given primary, generated by the action of some (as yet unknown) chiral algebra.

The partition function of the theory is then constructed from bilinears in the characters:

$$Z(\tau,\bar{\tau}) = \sum_{i,j=0}^{n-1} \bar{\chi}_i(\bar{\tau}) M_{ij} \chi_j(\tau) \qquad (4.2)$$

Here M_{Pij} is a constant matrix. In what follows, we confine our attention to the case when M_{ij} is diagonal*.

Under modular transformations, the characters transform as

* It is known that whenever it is possible to construct a modular-invariant partition function from a non-diagonal combination of characters, there also exists modular-invariant diagonal combination.

$$\chi_i(\tau+1)\, e^{2\pi i(h_i - \frac{c}{24})}$$

$$\chi_i\left(-\frac{1}{\tau}\right) = S_{ij}\chi_j(\tau) \qquad (4.3)$$

A modular-invariant diagonal partition function Z (Eq. (64)) will exist if S_{ij} leaves invatiant the matrix

$$M = \begin{pmatrix} 1 & & & & \\ & M_1 & & & \\ & & M_2 & & \\ & & & \ddots & \\ & & & & M_{n-1} \end{pmatrix} \qquad (4.4)$$

The numbers M_i appearing in the diagonal matrix M represent the number of distinct primaries in the theory with the same character.

The fusion rules of a conformal field theory are defined to be positive integers, N_{ijk} which count the number of distinct ways in which primaries i and j(or their descendants) can fuse to give the representation k.

Verlinde showed that, as matrices acting in the space of primary fields, the S_{ij} determine the fusion rules completely.

Finally we examine the power-series expansion of the characters. This must be of the form

$$\chi_i(\tau) = q^{h_i - \frac{c}{24}} \sum_{n=0}^{\infty} a_n^{(i)} q^n \qquad (4.5)$$

where h_i is the conformal dimension of the primary field above which the character is built. The important point here is that the coefficients $a_n^{(i)}$ must all be integers ≥ 0, simply because they count the number of states at each level.

S.Mukhi

V. CHERN-SIMONS THEORIES

The basic example seems to be Chern-Simons gauge theory. So we pick a compact gauge group G and a positive integer k. We introduce a gauge field $A_i^a(x)$ and write the Lagrangian

$$\mathcal{L} = \frac{k}{4\pi}\int_M Tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A) \qquad (5.1)$$

No metric is needed, so this is generally covariant.

The reason that the quantum field theory is exactly soluble is that this theory is trivial locally. For instance, at the classical level the Euler-Lagrange equation for this theory is $F_{ij}{}^a=0$, and implies that A_i can be gauged away locally. Thus, as in twistor theory, all information is to be encoded globally.

* Actually, the S_{ij} which satisfy Verlinde's identity act in the space of primaries, which is generically larger than the space of characters.

The latter spaces are essentially the Jones braid representation that are associated with the celebrated Jones polynomial of knot theory; the relation of these spaces to conformal field theory was first perceived by Tsuchiya and Kanie.

More concretely, though, Moore and Seiberg have recently shown that G/H models may also be interpreted as Chern-Simons theories of an appropriate group (essentially $G\times H$).

The only three dimensional generally covariant theories I know of that at first sight apear not to be Chern-Simons theories are general relativity and the three dimensional reduction of Donaldson theory. But both three dimensional gravity and the three dimensional reduction of Donaldson theory have turned out to be Chern-Simons theories. So one may conjecture *that* **every three dimensional generally covariant theory** is actually a Chern-Simons theory of some group of supergroup, not necessarily connected or simply connected.

In conclusion, if one considers rational conformal field theories and their cousins, integrable systems, in two dimensions, one sees that these are extremely rich systems. There are an incredible variety of extremely rich facts,related to each other in an incredible diversity of ways. To bring order to this chaos looks hopeless. But by stepping out of flatland and looking at things from the vantage point of three dimensions, one can find a more powerful viewpoint, where rational and integrable systems can be derived from a subtler and more incisive starting point. This step out of flatland, to a higher vantage point from which wider symmetry can be seen is temptingly akin to what we need in string theory. What is more, we have been urged to take yet another step to four dimensions, the most physcal dimension, the richest dimension for geometry, and the critical dimension for quantum field theory.

E.Witten

VI. DUALITY AND THE ROLE OF NONPERTURBATIVE EFFECTS ON THE WORLD-SHEET

Discussions of string theory at high temperatures duality enters in a crucial way giving a relation between physics at temperature T and $1/T$.

We shall actually see, the the physics of string beyond the Planck scale requires a new form of uncertainty relation as a consequence of duality.

The same is true for the case of torus-compactification even in the presence of arbitrary background fields. Self dual points in parameter space very often correspond to multicritical points of the underlying conformal field theory and also lead to an enhancement of the space-time gauge group.

The nonrenormalization theorems of supersymmetry guarantee that duality remains exact in any order of perturbation theory.

We here generalize duality to models obtained through orbifold compactification of the heterotic string. While it is relatively easy to see that the spectra of such theories are dual, it is not clear whether this also holds for the interactions.

In the following we shall show that this is not the case.

$$\frac{1}{4}m_L^2 = N_L - 1 + \frac{P_L^2}{2} + \frac{p^2}{2}$$
$$\frac{1}{4}m_R^2 = N_R + \frac{P_R^2}{2} \qquad (6.1)$$

$$P_{R,L} = \frac{1}{2}\frac{m}{R} \mp nR \qquad (6.2)$$

(we have used here the usual convention $\alpha' = 1/2$).

At this point there appear new massless states with $P_L^2 = 2$, $P_R^2 = 0$ and the symmetry is enhanced to $SU(2) \times E_8 \times E_8$.

Duality can be shown to be a good symmetry for heterotic string theories compactified on tori. This can be demonstrated including arbitary background values for the antisymmetric tensor fields as well as Wilson lines.

String theories on orbifolds split into two sectors: untwisted and twisted. The states in the untwisted sector belong to a subset of the states of the theory on the torus, namely those invariant under the twist. The states in the twisted sectors correspond to those strings which are closed on the orbifold but not on the torus. They can be obtained through a $\tau \to 1/\tau$ modular tranformation from the states in the untwisted sector.

A duality transformation exchanges p and w, where as the *twist* acts on p and w separately. For example in the Z_2 case we have as invariant states those created by

$$\bar{V}_{p,w} = \frac{1}{\sqrt{2}}(V_{p,w} + V_{-p,-w}) \qquad (6.3)$$

Recall, however, that modular transformation mix twisted and untwisted sector and therefore the intimate connection between duality and modular invariance as found on the torus does not seem to extend to orbifolds.

Let us illustrate our results in case of a simple example: two dimensions compactified on a torus defined by the SU(3) root lattice. The twist is chosen to be a $\frac{2\pi}{3}$ rotation. Our results can be easily generalized to six compacitifed dimensions.

As a basis of the lattice we choose (in complex notation).

$$e_1 = \sqrt{2} \quad e_2 = \frac{1}{\sqrt{2}} + i\sqrt{\frac{3}{2}}\,. \qquad (6.4)$$

while the dual lattice is given by

$$e^{*1} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{6}} \quad e^{*2} = i\sqrt{\frac{2}{3}} \qquad (6.5)$$

Winding vectors are defined by $w = n^i.e_iR$ while momentum vectors are given by $p = m_i.e^{*i}/R$ (with R denoting the radius of compactified space). On the torus p and w are conserved. In the presence of the Z_3 twist, this is no longer true, but a discrete subsymmetry remains. The twist identifies the generic winding vector $w = n^1 e1 + n^2 e_2$ with $-w = (n^1 - n^2)e_1 + n^1 e_2$. As a result we define a new winding number $N = (n^1 - n^2)\ mod 3$ which is invariant under the twist. Similarly we find $M = (m_1 + m_2) mod 3$ as the remnant of momentum conservation in compactified space after moding out by the twist.

The value of the three-point coupling $C_{p,w}$ is for the Z_3 orbifold:

$$C_{p,w} = \frac{\sqrt{3}}{\delta^{\frac{1}{2}(P_L^2 + P_R^2)}} \qquad (6.6)$$

with $\delta = 27$ and $P_{L,R}$ the Narain lattice vectors constructed with p and w.

Performing this rotation shows, however, that in the $\tilde{\sigma}$-basis **there is no way to uniquely associate relative winding number** N.

In conclusion we still can see that there is no way to distinguish between large and small R via trilinear couplings of the form $<\sigma\sigma V>$.

Before going any further, let us stress once more the fact that the existence of duality is a serious hint about a breakdown of our geometrical concepts for distances small compared to α'.

This is a new kind of uncertainty relation which is a consequence of duality even at the classical level.

The question of duality can now be rephrased in the following way: given two Yukawa couplings, one of order unity and one exponentially suppressed, can we conclude that the corresponding R is large compared to α'?

This identity is found to be

$$\tilde{Y}_i(R) = Y_i(\frac{1}{2\sqrt{3}R} \qquad (6.7)$$

and as a result of it we cannot distinguish between large and small R.

We have shown here the identity in the case of Z_3 in two dimensions but the result is easily generalized to Z_n, (N=4,6) in d=2 as well as to higher number of compactified dimensions. We also have restricted ourselves here to the case without Wilson lines.

Such an uncertainty relation is very similar to the one derived from string scatterings at ultra-Planckian energies and a unique feature of string theory. It shows that our naive geometrical picture of space-time breaks down beyond the Planck scale. It seems also to be at the origin of the fact that, for the motion of closed strings, orbifolds are as good objects as smooth manifolds. The appearance of duality does not seem to have anything to do with the existence of space-time supersymmetry although the nonrenormalization theorems of the latter seem to secure duality to any order of string perturbation theory. Ultimately such a symmetry could explain the existence of a minimal length.

It seems that, duality involves many concepts appearing naturally in string theories. Since duality is an inherently "stringy" symmetry it might reveal some properties of string theories that cannot be shared by conventional field theories of pointlike particles.

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